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SIMPLE LIE COLOR ALGEBRAS OF WEYL TYPE*

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ABSTRACT

A class of graded simple associative algebras are constructed, and from them, simple Lie color algebras are obtained. The structure of these simple Lie color algebras is explicitly described. More precisely, for an (ϵ, Γ)-color-commutative associative algebra A with an identity element over a field F of characteristic not 2, and for a color-commutative subalgebra D of color-derivations of A, denote by $A[D]$ the associative subalgebra of $\mathrm{End}(A)$ generated by A (regarded as operators on A via left multiplication) and D. It is easily proved that, as an associative algebra, $A[D]$ is Γ-graded simple if and only if A is Γ-graded D-simple. Suppose

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A is Γ-graded D-simple. Then, (a) $A[D]$ is a free left A-module; (b) as a Lie color algebra, the subquotient $[A[D],A[D]]/Z(A[D]) \cap [A[D],A[D]]$ is simple (except one minor case), where $Z(A[D])$ is the color center of $A[D]$.

1. Introduction

Lie color algebras are generalizations of Lie superalgebras. Let us start with the definition. Let F be a field, Γ an additive group. A **skew-symmetric bicharacter** of Γ is a map $\epsilon\colon \Gamma \times \Gamma \to F^*$ satisfying

$$\epsilon(\lambda,\mu)=\epsilon(\mu,\lambda)^{-1}, \quad \epsilon(\lambda,\mu+\nu)=\epsilon(\lambda,\mu)\epsilon(\lambda,\nu), \quad \forall \lambda,\mu,\nu \in \Gamma. \tag{1.1}$$

It is clear that $\epsilon(\lambda,0)=1$ for any $\lambda \in \Gamma$. Let $L=\bigoplus_{\lambda\in\Gamma} L_\lambda$ be a Γ-graded F-vector space. For a nonzero homogeneous element $a \in L$, denote by $\bar{a}$ the unique group element in Γ such that $a \in L_{\bar{a}}$. We shall call $\bar{a}$ the **color** of a. The F-bilinear map $[\cdot,\cdot]\colon L\times L \to L$ is called a Lie color bracket on L if the following conditions are satisfied:

$$[a,b]=-\epsilon(\bar{a},\bar{b})[b,a], \quad \text{(skew symmetry)} \tag{1.2}$$

$$[a,[b,c]]=[[a,b],c]+\epsilon(\bar{a},\bar{b})[b,[a,c]], \quad \text{(Jacoby identity)} \tag{1.3}$$

for all homogeneous elements $a,b,c \in L$. The algebra structure $(L,[\cdot,\cdot])$ is called an (ϵ,Γ)**-Lie color algebra** or simply a **Lie color algebra**. If $\Gamma=\mathbb{Z}/2\mathbb{Z}$ and $\epsilon(i,j)=(-1)^{ij}, \forall i,\ j \in \mathbb{Z}/2\mathbb{Z}$, then (ϵ,Γ)-Lie color algebras are simply Lie superalgebras. For Lie color algebras, we refer the reader to [P2] and the references there. This paper constructs a class of simple Lie color algebras, and explicitly describes the structure of these simple Lie color algebras.

Let $A=\bigoplus_{\lambda\in\Gamma} A_\lambda$ be a Γ-graded associative F-algebra with an identity element 1, i.e., $A_\lambda A_\mu \subset A_{\lambda+\mu}$ for all $\lambda,\mu \in \Gamma$. So $1 \in A_0$. We say that A is **graded simple** if A does not have nontrivial Γ-graded ideals. Denote by $H(A)$ all homogeneous elements of A. If we define the bilinear product $[\cdot,\cdot]$ on A by

$$[x,y]=xy-\epsilon(\bar{x},\bar{y})yx, \quad \forall x,y \in H(A), \tag{1.4}$$

then $(A,[\cdot,\cdot])$ becomes a (ϵ,Γ)-Lie color algebra. We shall simply write $\epsilon(x,y)$ for $\epsilon(\bar{x},\bar{y})$ for $x,y \in H(A)$.

A **Lie color ideal** U of A is a Γ-graded vector space U of A such that $[A, U] \subset U$. Sometimes it is called an (ϵ, Γ)-Lie ideal, or simply a color ideal. The **ϵ-center** $Z(A)$ of A is defined as

$$Z = Z(A) = \{x \in A | [x, A] = 0\}. \tag{1.5}$$

It is easy to see that $Z(A)$ is Γ-graded. We say that A is **color-commutative** (or ϵ-color commutative) if $[A, A] = 0$.

Passman [P1, P2] proved that, for a color-commutative associative algebra A with an identity element over a field F, and for a color-commutative color-derivation subalgebra D of A, the Lie color algebra (including the Lie algebra case) $AD = A \odot D$ is simple if and only if A is graded D-simple and AD acts faithfully on A, and $\operatorname{char} F \neq 2$ or $\dim_F D \geq 2$ or $D(A) = A$. In [SZ1, SZ2, Z1], (associative and Lie) algebras of Weyl type were constructed and studied. In this paper we shall study the color version of these algebras.

For a color-commutative associative algebra A with an identity element over a field F of characteristic not 2, and for a color-commutative subalgebra D of color-derivations of A, denote by $A[D]$ the associative subalgebra of $\operatorname{End}(A)$ generated by A (regarding as operators on A via multiplication) and D. It is easily proved that, as an associative algebra, $A[D]$ is graded simple if and only if A is graded D-simple (see Theorem 2.2). Suppose A is graded D-simple. Then, (a) $A[D]$ is a free left A-module (Theorem 3.2); (b) as a Lie color algebra, the subquotient $[A[D], A[D]]/Z(A[D]) \cap [A[D], A[D]]$ is simple (except one minor case), where $Z(A[D])$ is the ϵ-center of $A[D]$. The structure of this subquotient is explicitly described (see Theorem 3.9). In many cases $A[D] = [A[D], A[D]]$.

2. Graded simple associative algebras of Weyl type

Throughout this work, we assume that A is Γ-graded and (ϵ, Γ)-color commutative and the field F is of characteristic not 2. An F-linear transformation $\partial: A \to A$ is called a **homogeneous color-derivation** of degree $\bar{\partial} \in \Gamma$ if

$$\begin{aligned} &\partial(a) \in A_{\bar{\partial}+\bar{a}}, \quad \forall a \in H(A) \quad \text{and} \\ &\partial(ab) = \partial(a)b + \epsilon(\partial, a)a\partial(b), \quad \forall a, b \in H(A), \end{aligned} \tag{2.1}$$

where we simply denote $\epsilon(\bar{\partial}, \bar{a})$ by $\epsilon(\partial, a)$.

Taking $a = b = 1$ in (2.1), we obtain $\partial(1) = 0$ and so $\partial(c) = 0$ for all $c \in F$. Denote $\operatorname{Der}^\epsilon(A) = \bigoplus_{\lambda \in \Gamma} \operatorname{Der}^\epsilon_\lambda(A)$, where $\operatorname{Der}^\epsilon_\lambda(A)$ is the F-vector space spanned by all homogeneous color derivations of degree $\lambda \in \Gamma$. As in the Lie algebras case, it is easy to verify that $\mathrm{Der}^{\epsilon}(A)$ becomes an (ϵ,Γ)-Lie color algebra under the Lie color bracket

$$[\partial,\partial'] = \partial\partial' - \epsilon(\partial,\partial')\partial'\partial, \quad \forall \partial,\partial' \in H(\mathrm{Der}^{\epsilon}(A)), \tag{2.2}$$

where $\partial\partial'$ is the composition of the operators ∂ and ∂'.

Let $D = \bigoplus_{\lambda\in\Gamma} D_\lambda$ be a nonzero Γ-graded color-commutative Lie color subalgebra of $\mathrm{Der}^{\epsilon}(A)$, i.e.,

$$\partial\partial' = \epsilon(\partial,\partial')\partial'\partial, \quad \forall \partial,\partial' \in H(D). \tag{2.3}$$

We call A **graded D-simple** if A has no nontrivial graded D-stable ideals (see [P2]). Set $\Gamma_+ = \{\lambda \in \Gamma | \epsilon(\lambda,\lambda) = 1\}$. Then by (1.1), Γ_+ is a subgroup of Γ with index ≤ 2. Set $\Gamma_- = \{\lambda \in \Gamma | \epsilon(\lambda,\lambda) = -1\}$. For any graded subspace B of A, we define $B_+ = \bigoplus_{\lambda\in\Gamma_+} B_\lambda$; then B_+ is Γ-graded. Similarly we can define B_-. Since $\Gamma = \Gamma_+ \cup \Gamma_-$, it follows that $B = B_+ \oplus B_-$. By (2.3), we have $\partial^2 = 0$ if $\partial \in H(D_-)$.

Fix a homogeneous basis $\{\partial_i | i \in I\}$ of D, where I is some index set. Fix a total ordering $<$ on I. Define J to be the set of all $\alpha = (\alpha_i | i \in I) \in \mathbb{Z}_+^I$ such that $\alpha_i = 0$ for all but a finite number of $i \in I$ and $\alpha_i = 0$ or 1 if $\bar{\partial}_i \in \Gamma_-$. For $\alpha \in J$, we define the **support** of α by $\mathrm{supp}(\alpha) = \{i \in I | \alpha_i \neq 0\}$ and define the **level** of α by $l(\alpha) = |\alpha| = \sum_{i\in I} \alpha_i$. Define a total ordering on J by

$$\begin{aligned} \alpha < \beta \Leftrightarrow |\alpha| < |\beta|, \quad \text{or } |\alpha| = |\beta| \quad \text{and } \exists i \in I \text{ such that} \\ \alpha_i < \beta_i \text{ and } \alpha_j = \beta_j, \forall j \in I, j < i. \end{aligned} \tag{2.4}$$

Denote by $F[D]$ the Γ-graded associative color-commutative algebra generated by D, and denote by $A[D]$ the associative subalgebra of $\mathrm{End}(A)$ generated by A (regarded as operators on A via left multiplication) and D. For $x \in A, \alpha \in J$, we define by $x\partial^\alpha$ the operator acting on A, i.e.,

$$(x\partial^\alpha)(y) = x(\partial_{i_1}^{\alpha_{i_1}}(\partial_{i_2}^{\alpha_{i_2}} \cdots (\partial_{i_r}^{\alpha_{i_r}}(y)) \cdots)), \quad \forall y \in A, \tag{2.5}$$

where $\partial^\alpha = \partial_{i_1}^{\alpha_{i_1}}\partial_{i_2}^{\alpha_{i_2}} \cdots \partial_{i_r}^{\alpha_{i_r}}$ with $i_1 < i_2 < \cdots < i_r$. We shall call α the **degree** of ∂^α. Define $\epsilon^+(\alpha,\beta)$ for $\alpha,\beta \in J$ by $\partial^\alpha\partial^\beta = \epsilon^+(\alpha,\beta)\partial^{\alpha+\beta}$; then

$$\epsilon^+(\alpha,\beta) = \prod_{i,j\in I: i>j} \epsilon(\partial_i,\partial_j)^{\alpha_i\beta_j}, \quad \forall \alpha,\beta \in J. \tag{2.6}$$

We shall also simply write $\epsilon^+(x,y)$ for $\epsilon^+(\bar{x},\bar{y})$ for $x, y \in H(A[D])$. From $\epsilon^+(\alpha,\beta)\partial^{\alpha+\beta} = \partial^\alpha\partial^\beta = \epsilon(\alpha,\beta)\partial^\beta\partial^\alpha = \epsilon(\alpha,\beta)\epsilon^+(\beta,\alpha)\partial^{\alpha+\beta}$, we see that

$$\epsilon(\alpha,\beta) = \epsilon^+(\alpha,\beta)\epsilon^+(\beta,\alpha)^{-1}, \quad \forall \alpha,\beta \in J, \tag{2.7}$$

If $\alpha+\beta \notin J$, we set $\partial^{\alpha+\beta}=0$. For any $\alpha \in J$, set

$$J(\alpha)=\{\gamma \in J | \gamma_i \leq \alpha_i,\ \forall i \in I\}. \tag{2.8}$$

Denote $\binom{\alpha}{\gamma}=\prod_{i\in I}\binom{\alpha_i}{\gamma_i}$, $\forall \alpha,\gamma \in J$. Then $\binom{\alpha}{\gamma}=0$ if $\gamma \notin J(\alpha)$. We have

$$\partial^{\alpha}(xy)=\sum_{\gamma\in J}\binom{\alpha}{\gamma}\epsilon^{+}(\alpha-\gamma,\gamma)^{-1}\epsilon(\partial^{\gamma},x)\partial^{\alpha-\gamma}(x)\partial^{\gamma}(y), \tag{2.9}$$
$$\forall \alpha \in J,\ x,y \in H(A).$$

One may easily check that

$$(u\partial^{\alpha})\cdot(v\partial^{\beta})=\sum_{\lambda\in J}\binom{\alpha}{\lambda}\epsilon^{+}(\alpha-\lambda,\lambda)^{-1}\epsilon(\partial^{\lambda},v)\epsilon^{+}(\lambda,\beta)u\partial^{\alpha-\lambda}(v)\partial^{\beta+\lambda}, \tag{2.10}$$
$$\forall u,v \in H(A),\alpha,\beta \in J.$$

Clearly, formula (2.10) defines a Γ-graded associative algebra $(A[D],\cdot)$ such that A is a left $A[D]$-module via (2.5). For any

$$x=\sum_{\alpha\in J}u_{\alpha}\partial^{\alpha} \in A[D], \quad u_{\alpha}\in A, \tag{2.11}$$

the expression in (2.11) is sometimes not unique. An expression of x in (2.11) is called **principal**, if the integer $\max\{|\alpha| \mid u_{\alpha}\neq 0\}$ is minimal. We denote this integer by $h(x)$. Set

$$F_1=A^{D}=\{u\in A|D(u)=0\}. \tag{2.12}$$

From Lemma 2.1 in [P2], any nonzero $a\in H(F_1)$ is invertible (i.e., F_1 is a graded field) when A is **Γ-graded D-simple**. In this case $(F_1)_-=0$.

LEMMA 2.1: (i) $A\cap A[D]D=0$.
(ii) *The ϵ-center $Z(A[D])$ of $A[D]$ is F_1.*

Proof: Note that we have assumed that A is Γ-graded (ϵ,Γ)-commutative. Part (i) follows from the action of $x\in A\cap A[D]D$ on 1.

(ii) For all $u\in H(F_1)$ and $\delta\in H(D)$, we have $[\delta,u]=\delta(u)=0$. So $F_1\subset Z(A[D])$. Suppose $x\in H(Z(A[D]))$. If $h(x)>0$, write $x=x_0+x_1$ where $x_0\in H(A)$ and $x_1\in H(A[D]D)$. Then $x_1\neq 0$. Choose $a\in H(A)$ such that $x_1(a)\neq 0$. It follows that $0=[x,a]=[x_1,a]=x_1(a)+y$, where $y\in A[D]D$. From (i) we deduce that $x_1(a)=y=0$, a contradiction. So $h(x)=0$, i.e., $x\in A$. We have

$$x\partial=\epsilon(\bar{x},\bar{\partial})\partial x=\epsilon(\bar{x},\bar{\partial})\partial(x)+x\partial,\quad \forall\partial\in H(D),$$

to give $\partial(x) = 0$ for all $\partial \in H(D)$, i.e., $x \in F_1$. Therefore Lemma 2.1 follows. ∎

THEOREM 2.2: *Suppose that A is a Γ-graded (ϵ, Γ)-commutative F-algebra with an identity, and $D \subset Der^\epsilon(A)$ is a color commutative subspace. Then the Γ-graded associative algebra $A[D]$ is graded simple if and only if A is Γ-graded D-simple.*

Proof: "$\Rightarrow$": Suppose A is not graded D-simple. Choose a nonzero proper graded D-stable ideal $\mathcal{K}$. Then clearly $\mathcal{K}[D]$ is a nonzero graded color ideal of $A[D]$. Since $A[D]$ is graded simple, then $\mathcal{K}[D] = A[D]$, in particular, $A \subset \mathcal{K}[D]$. From Lemma 2.1 (i), we know that $A \subset \mathcal{K}$, a contradiction. Thus A is graded D-simple.

"$\Leftarrow$": Suppose L is a nonzero graded ideal of $A[D]$. It suffices to show that $L = A[D]$.

Suppose that $L \cap A = 0$. Choose $x \in H(L)\backslash\{0\}$ such that $h(x)$ is minimal. So $h(x) > 0$. Write $x = x_0 + x_1$ where $x_0 \in H(A)$ and $x_1 \in H(A[D]D)$. Then $x_1 \neq 0$. Choose $a \in H(A)$ such that $x_1(a) \neq 0$. It follows that

$$x' = [x, a] = [x_1, a] = x_1(a) + y \in L,$$

where $y \in H(A[D]D)$. From (1.4) and the computation (2.10), we deduce that $h(x') = h(y) < h(x)$. By the minimality of $h(x)$ we deduce that $x' = 0$. Applying Lemma 2.2 (i) gives $x_1(a) = y = 0$, a contradiction. Thus $L \cap A \neq 0$. It is clear that $L \cap A$ is a graded D-ideal of A. Since A is graded D-simple, $L \cap A = A$. In particular $1 \in L$. Therefore $L = A[D]$. ∎

3. Simple Lie color algebras of Weyl type

In this section we assume that A is a Γ-graded D-simple and (ϵ, Γ)-commutative associative algebra with an identity, the field F is of characteristic not 2, and $D \subset \mathrm{Der}^\epsilon(A)$ is color commutative. In this section we shall first study the structure of $A[D]$ as a left A-module, then investigate the Lie structure of $A[D]$. We still use the notation in Section 2.

LEMMA 3.1: (i) *If $x\partial^\alpha = 0$ for some $x \in H(A)\backslash\{0\}$ and some $\alpha \in J$, then $\partial^\alpha = 0$.*

(ii) $A \subset [A[D], A[D]]$.

Proof: (i) Let $C = \{x \in A | x\partial^\alpha = 0\}$. It is easy to verify that C is a nonzero Γ-graded D-ideal of A. Since A is Γ-graded D-simple, then $C = A$. It yields that $\partial^\alpha = 0$.

(ii) From $[x\partial, y] = x\partial(y) \in [A[D], A[D]]$ for all $x, y \in H(A)$ and $\partial \in H(D)$, we obtain that $AD(A) \subset [A[D], A[D]]$. Since $AD(A)$ is a nonzero Γ-graded D-ideal of A and A is Γ-graded D-simple, $A = AD(A)$. Thus $A \subset [A[D], A[D]]$. ∎

If $\operatorname{char} F = p > 0$, observe that for any $\partial \in \operatorname{Der}^\epsilon(A)$, one has $\partial^p \in \operatorname{Der}^\epsilon(A)$. For convenience, no matter whether $\operatorname{char} F = p > 0$ or $\operatorname{char} F = 0$, we denote

$$\mathcal{D} = \operatorname{Der}^\epsilon(A) \cap F_1[D], \tag{3.1}$$

where $F_1[D]$ is the subalgebra of $\operatorname{End}(A)$ generated by F_1 and D. Let $\{d_i | i \in \mathcal{I}\}$ be a homogeneous F_1-basis for $\mathcal{D}$, where $\mathcal{I}$ is some index set. It is clear that $D \subset \mathcal{D}$, and that A is graded D-simple if and only if A is graded $\mathcal{D}$-simple. Let

$$\begin{aligned}\mathcal{J} = \{\alpha = (\alpha_i | i \in \mathcal{I}) |\ & \alpha_i \in \mathbb{Z}_+, \text{ and } \alpha_i \le p-1 \text{ if } \operatorname{char} F = p > 0,\\ & \text{and } \alpha_i = 0 \text{ or } 1 \text{ if } \bar{d}_i \in \Gamma_-,\\ & \text{and } \alpha_i = 0 \text{ for all but a finite number of } i \in \mathcal{I}\}.\end{aligned} \tag{3.2}$$

We will often simply denote $\alpha = (\alpha_i | i \in \mathcal{I})$ by $\alpha = (\alpha_i)$. We also fix a total ordering $<$ on $\mathcal{I}$ and define a total ordering on $\mathcal{J}$ as in (2.7). We also write $d^\alpha = \prod_{i \in I} d^{\alpha_i}$ according to the ordering $<$ on $\mathcal{I}$ for $\alpha \in \mathcal{J}$. Then

$$A[D] = \sum_{\alpha \in \mathcal{J}} A d^\alpha. \tag{3.3}$$

THEOREM 3.2: *Suppose that the field F is of characteristic not 2, A is a Γ-graded D-simple and (ϵ, Γ)-commutative associative F-algebra with an identity, where $D \subset Der^\epsilon(A)$ is a color commutative subspace. Then $A[D]$ is a free A-module with the homogeneous basis $\{d^\alpha | \alpha \in \mathcal{J}\}$.*

We break the proof of this theorem into several Lemmas. For $\mathcal{J}_0 \subset \mathcal{J}, B \subset A$, we say that the sum $\sum_{\alpha \in \mathcal{J}_0} B d^\alpha$ is **direct** if a finite sum $\sum_{\alpha \in \mathcal{J}_0} b_\alpha d^\alpha = 0$ with $b_\alpha \in B$ implies $b_\alpha d^\alpha = 0$ for all $\alpha \in \mathcal{J}_0$.

LEMMA 3.3: *Let $\mathcal{J}_0 \subset \mathcal{J}$. The sum $\sum_{\alpha \in \mathcal{J}_0} A d^\alpha$ is direct if and only if the sum $\sum_{\alpha \in \mathcal{J}_0} F_1 d^\alpha$ is direct.*

Proof: "$\Rightarrow$": This direction is clear.

"$\Leftarrow$": Suppose $\sum_{\alpha \in \mathcal{J}_0} A d^\alpha$ is not direct. There exists $\mathcal{J}_1 = \{\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(r)}\} \subset \mathcal{J}_0$ with $|\mathcal{J}_1| > 1$ such that $A d^{\alpha^{(0)}} \cap \sum_{i=1}^r A d^{\alpha^{(i)}} \neq 0$. Choose such a minimal

r. Let

$$ud^{\alpha^{(0)}} = \sum_{i=1}^{r} u_i d^{\alpha^{(i)}} \in \left(Ad^{\alpha^{(0)}} \cap \sum_{i=1}^{r} Ad^{\alpha^{(i)}}\right)\backslash\{0\}, \tag{3.4}$$

for some $u, u_i \in H(A)$, and let $I_1 = \{0, 1, \ldots, r\}$. It follows that $d^{\alpha^{(i)}} \neq 0$ for any $i \in I_1$. Let

$$B_0 = \left\{x \in A | xd^{\alpha^{(0)}} \in \sum_{i \in I_1 \backslash \{0\}} Ad^{\alpha^{(i)}}\right\}.$$

Then it is easy to see that B_0 is a nonzero Γ-graded D-ideal of A. Since A is Γ-graded D-simple, then $B_0 = A$. Then

$$d^{\alpha^{(0)}} = \sum_{i \in I_1 \backslash \{0\}} a_i d^{\alpha^{(i)}} \quad \text{for some } a_i \in A. \tag{3.5}$$

By taking bracket with $\partial \in H(\mathcal{D})$, we deduce that

$$0 = [\partial, d^{\alpha^{(0)}}] = \left[\partial, \sum_{i \in I_1 \backslash \{0\}} a_i d^{\alpha^{(i)}}\right] = \sum_{i \in I_1 \backslash \{0\}} \partial(a_i) d^{\alpha^{(i)}}.$$

From the minimality of r, we must have

$$\partial(a_i) d^{\alpha^{(i)}} = 0, \quad \forall \partial \in \mathcal{D},\ 1 \leq i \leq r. \tag{3.6}$$

Suppose $D(a_i) \neq 0$ for some $i \in I_1$, say $D(a_1) \neq 0$. Lemma 3.1 yields that $d^{\alpha^{(1)}} = 0$, a contradiction. Thus $D(a_i) = 0$ for all $i \in I_1$, i.e., $a_i \in F_1$ in (3.5). Therefore the sum $\sum_{\alpha \in \mathcal{J}_1} F_1 d^\alpha$ is not direct. This proves the lemma. ∎

Lemma 3.4: $A[D] = \bigoplus_{\alpha \in \mathcal{J}} Ad^\alpha$.

Proof: Suppose the sum in (3.3) is not direct. From Lemmas 2.1 (i) and 3.3, there exists $\mathcal{J}_1 = \{\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(r)}\} \subset \mathcal{J}\backslash\{0\}$ such that $F_1 d^{\alpha^{(0)}} \cap \sum_{i=1}^{r} F_1 d^{\alpha^{(i)}} \neq 0$. Let $I_1 = \{0, 1, \ldots, r\}$, and let $h(\mathcal{J}_1) = \max\{|\alpha^{(i)}| | i \in I_1\}$. Then $h(\mathcal{J}_1) > 1$. We choose $\mathcal{J}_1$ such that $h(\mathcal{J}_1)$ is minimal, and then r is minimal. Let

$$-d^{\alpha^{(0)}} = \sum_{i=1}^{r} a_i d^{\alpha^{(i)}} \in \left(F_1 d^{\alpha^{(0)}} \cap \sum_{i=1}^{r} F_1 d^{\alpha^{(i)}}\right)\backslash\{0\}, \tag{3.7}$$

for some $a_i \in H(F_1)$, $1 \leq i \leq r$. Denote $a_0 = 1$. It follows that $a_i d^{\alpha^{(i)}} \neq 0$ for $i \in I_1$. Rewrite (3.7) to give $\sum_{i=0}^{r} a_i d^{\alpha^{(i)}} = 0$. Then for $x \in H(A)$, $\sum_{i=0}^{r} a_i d^{\alpha^{(i)}}(x) =$

0. By (2.10),

$$0=\left[\sum_{i=0}^{r}a_id^{\alpha^{(i)}},x\right]=\sum_{i=0}^{r}\sum_{\gamma\in\mathcal{J}}a_i\binom{\alpha^{(i)}}{\gamma}\epsilon^+(\alpha^{(i)}-\gamma,\gamma)^{-1}\epsilon(d^\gamma,x)d^{\alpha^{(i)}-\gamma}(x)d^\gamma$$
$$=\sum_{\gamma\in\mathcal{J},0<|\gamma|<h(\mathcal{J}_1)}\left(\sum_{i=0}^{r}a_i\binom{\alpha^{(i)}}{\gamma}\epsilon^+(\alpha^{(i)}-\gamma,\gamma)^{-1}\epsilon(d^\gamma,x)d^{\alpha^{(i)}-\gamma}(x)\right)d^\gamma.$$

From the minimality of $h(\mathcal{J}_1)$ and Lemma 2.1(i), it follows that

$$\sum_{i=0}^{r}a_i\binom{\alpha^{(i)}}{\gamma}\epsilon^+(\alpha^{(i)}-\gamma,\gamma)^{-1}\epsilon(d^\gamma,x)d^{\alpha^{(i)}-\gamma}(x)d^\gamma=0, \tag{3.8}$$

for all $x\in H(A)$ and all $\gamma\in\mathcal{J}\backslash\{0\}$ with $0<|\gamma|<h(\mathcal{J}_1)$. Note that for any $\gamma\in\mathcal{J}$ with $\alpha^{(i)}-\gamma\in\mathcal{J}$, we have $d^\gamma d^{\alpha^{(i)}-\gamma}=\epsilon(d^\gamma,d^{\alpha^{(i)}-\gamma})d^{\alpha^{(i)}}\neq 0$, to give $d^\gamma\neq 0$. Since $d^\gamma\neq 0$, from (3.8) and Lemma 3.1 we obtain

$$\sum_{i=0}^{r}a_i\binom{\alpha^{(i)}}{\gamma}\epsilon^+(\alpha^{(i)}-\gamma,\gamma)^{-1}\epsilon(d^\gamma,x)d^{\alpha^{(i)}-\gamma}(x)=0,$$

i.e.,

$$\sum_{i=0}^{r}a_i\binom{\alpha^{(i)}}{\gamma}\epsilon^+(\alpha^{(i)}-\gamma,\gamma)^{-1}d^{\alpha^{(i)}-\gamma}(\epsilon(d^\gamma,x)x)=0,$$

for all $x\in H(A)$ and all $\gamma\in\mathcal{J}$ with with $\alpha^{(i)}-\gamma\in\mathcal{J}$. So

$$\sum_{i=0}^{r}a_i\binom{\alpha^{(i)}}{\gamma}\epsilon^+(\alpha^{(i)}-\gamma,\gamma)^{-1}d^{\alpha^{(i)}-\gamma}=0$$

for all $\gamma\in\mathcal{J}$ with $0<h(\gamma)<h(\mathcal{J}_1)$. We have $a_i\binom{\alpha^{(i)}}{\gamma}\neq 0$ since $\alpha_j^{(i)}\leq p-1$ if $\operatorname{char}F=p>0$, for $\gamma\in\mathcal{J}$ with $\alpha^{(i)}-\gamma\in\mathcal{J}$. Thus we see that $d^{\alpha^{(i)}-\gamma}=0$ for all $\gamma\in\mathcal{J}\backslash\{0\}$ with $\alpha^{(i)}-\gamma\in\mathcal{J}\backslash\{0\}$. As we noted also $d^{\alpha^{(i)}-\gamma}\neq 0$, so this contradicts the minimality of $h(\mathcal{J}_1)$. Therefore $A[D]=\bigoplus_{\alpha\in\mathcal{J}_0}Ad^\alpha$. ∎

LEMMA 3.5: *For all $\alpha\in\mathcal{J}$, $d^\alpha\neq 0$.*

Proof: It is obvious that $d^\alpha\neq 0$ for $\alpha\in\mathcal{J}$ with $|\alpha|\leq 1$. Suppose $d^\alpha=0$ for some $\alpha\in\mathcal{J}$ with $|\alpha|\geq 2$. Choose α such that $|\alpha|$ is minimal. Then for any $x\in H(A)$, $d^\alpha(x)=0$. Then by (1.4) and (2.10),

$$0=[d^\alpha,x]=\sum_{\beta\in\mathcal{J}}\binom{\alpha}{\beta}\epsilon^+(\alpha-\beta,\beta)^{-1}\epsilon(\beta,\bar{x})d^{\alpha-\beta}(x)d^\beta,\quad\forall x\in H(A).$$

Applying Lemma 3.4 and noting that $\binom{\alpha}{\beta}\epsilon^+(\alpha-\beta,\beta)^{-1}\epsilon(\beta,\bar{x}) \neq 0$ (by noting that $\alpha_i \leq p-1$ if $\operatorname{char} F = p > 0$) for any $\beta \in \mathcal{J}$ with $\alpha - \beta \in \mathcal{J}$, we get $d^{\alpha-\beta}(x)d^{\beta} = 0, \forall x \in H(A), \beta \in \mathcal{J}$. Noting that $d^{\beta} \neq 0$ for any $\beta \in \mathcal{J}$ with $\alpha - \gamma \in \mathcal{J}\backslash\{0\}$ (since $|\alpha|$ is minimal), applying Lemma 3.1 gives $d^{\alpha-\beta}(x) = 0, \forall x \in H(A)$, i.e., $d^{\alpha-\beta} = 0$ for $\beta \in \mathcal{J}$ with $\alpha - \gamma \in \mathcal{J}\backslash\{0\}$, contradicting the minimality of $|\alpha|$. Therefore the assertion follows. ∎

Proof of Theorem 3.2: It follows from Lemmas 3.1, 3.3, 3.4 and 3.5. ∎

Denote by $W = [A[D], A[D]]$ the derived Lie color ideal of $A[D]$. In the rest of this section we will investigate the structure of W.

LEMMA 3.6: *Suppose $\dim_{F_1} \mathcal{D} > 1$ or $\mathcal{D} \neq \mathcal{D}_-$. Then for any homogeneous color-derivation $\partial \in \mathcal{D}$, we have $A\partial \subset W$.*

Proof:

CASE 1: $\partial \in \mathcal{D}_+$.

For any $x, a \in H(A)$, by Lemma 3.1(ii), $a, x\partial^2(a) \in W$, thus

$$x\epsilon(\partial, a)\partial(a)\partial = \frac{1}{2}([x\partial^2, a] - x\partial^2(a)) \in W, \tag{3.9}$$

i.e., $A\partial(A)\partial \subset W$. But the space $K = \{x \in A | x\partial \in W\}$ is Γ-graded D-stable since W is a Lie color ideal of $A[D]$, and K contains the nonzero graded ideal $A\partial(A)$ of A, by Lemma 3.1 in [P2], $K = A$, i.e., $A\partial \subset W$.

CASE 2: $\partial \in \mathcal{D}_-$.

First assume that $\mathcal{D}_+ \neq 0$. Choose $\partial' \in H(\mathcal{D}_+)\backslash\{0\}$; by Case 1, $A\partial' \subset W$. Thus for any $x, y \in H(A)$, we have $[x\partial', y\partial] \in W$ and

$$[x\partial', y\partial] = x\partial'(y)\partial - \epsilon(x\partial', y\partial)y\partial(x)\partial' \equiv x\partial'(y)\partial \pmod W, \tag{3.10}$$

i.e., $A\partial'(A)\partial \subset W$, thus as in Case 1, $A\partial \subset W$. Next assume that $\mathcal{D}_+ = 0$; then by the assumption of the lemma, we can choose $\partial' \in \mathcal{D}_-$ such that ∂, ∂' are F_1-linear independent. For $x, y \in H(A)$, we have $[x\partial', y\partial] \in W$ and

$$\begin{aligned}
[x\partial', y\partial] &= x\partial'(y)\partial - \epsilon(x\partial', y\partial)y\partial(x)\partial' \\
&= x\partial'(y)\partial - \epsilon(x\partial', y\partial)\epsilon(\partial, y)^{-1}(\partial(yx)\partial' - \partial(y)x\partial') \\
&\equiv x(\partial'(y)\partial + \epsilon(x\partial', y\partial)\epsilon(y, \partial)\epsilon(\partial(y), x)\partial(y)\partial') \pmod W \\
&= x(\partial'(y)\partial + \epsilon(\partial', y\partial)\epsilon(y, \partial)\partial(y)\partial'),
\end{aligned} \tag{3.11}$$

where the equality "$\equiv$" follows from the fact that $\partial(xy)\partial' = [\partial, xy\partial'] \in W$. Denote $d = \partial'(y)\partial + \epsilon(\partial', y\partial)\epsilon(y,\partial)\partial(y)\partial'$. Note that (3.11) shows that $Ad \in W$. By using $\partial^2 = 0$ and applying $\operatorname{ad}\partial$ to (3.11), we obtain $[\partial, xd] \in W$ and

$$[\partial, xd] = \partial(x)d + \epsilon(\partial, x)x[\partial, d] \equiv \epsilon(\partial, x)x\partial(\partial'(y))\partial \ (\operatorname{mod} W).$$

This shows that $A\partial(\partial'(A))\partial \subset W$. But $\partial\partial' \neq 0$ by Lemma 3.5; thus, as in Case 1, $A\partial \subset W$. ∎

For any $i \in \mathcal{I}$, we define

$$\delta^{(i)} \in \mathcal{J} \quad \text{such that } \delta_j^{(i)} = \delta_{i,j}, \quad \forall j \in \mathcal{I}. \tag{3.12}$$

The following technical lemma plays a crucial role in describing the structure of W.

Lemma 3.7: *Let $\beta \in \mathcal{J}$ with $\operatorname{supp}(\beta) = \{1, 2, \ldots, n\}$ and let $\partial_1', \partial_2', \ldots, \partial_n' \in \operatorname{Der}^\epsilon(A)$ be A-linear independent homogeneous derivations such that $\bar{\partial}_i' - \bar{d}_i = \bar{\partial}_j' - \bar{d}_j$ for $i, j = 1, 2, \ldots, n$. Suppose there exist $a_1, a_2, \ldots, a_n \in F\backslash\{0\}$ such that*

$$x\sum_{i=1}^{n} a_i\epsilon(d^{\beta-\delta^{(i)}}, b)\partial_i'(b)d^{\beta-\delta^{(i)}} \in W, \quad \forall x, b \in H(A); \tag{3.13}$$

then $Ad^{\beta-\delta^{(i)}} \subset W$ for $i = 1, 2, \ldots, n$.

Proof: By shifting the index, it suffices to prove $Ad^{\beta-\delta^{(n)}} \subset W$. We shall employ induction on n. If $n = 1$, (3.13) shows that $A\partial_1'(A)d^{\beta-\delta^{(n)}} \subset W$, thus the result follows from Lemma 3.1 in [P2]. Suppose $n \geq 2$. Replacing x by $x\partial_1'(a)$, and replacing x, b by $y\partial_1'(b), a$ in (3.13), we obtain respectively

$$\begin{aligned} &x\sum_{i=1}^{n} a_i\epsilon(d^{\beta-\delta^{(i)}}, b)\partial_1'(a)\partial_i'(b)d^{\beta-\delta^{(i)}} \in W, \\ &y\sum_{i=1}^{n} a_i\epsilon(d^{\beta-\delta^{(i)}}, a)\epsilon(\partial_1'(b), \partial_i'(a))\partial_i'(a)\partial_1'(b)d^{\beta-\delta^{(i)}} \in W. \end{aligned} \tag{3.14}$$

Setting $y = x\epsilon(d^{\beta-\delta^{(1)}}, a)^{-1}\epsilon(d^{\beta-\delta^{(1)}}, b)\epsilon(\partial_1'(b), \partial_1'(a))^{-1}$ and subtracting the two expressions of (3.14), we obtain

$$x\sum_{i=2}^{n} a_i\epsilon(d^{\beta-\delta^{(i)}}, b)(\partial_1'(a)\partial_i'(b) - u_i(a)\partial_i'(a)\partial_1'(b))d^{\beta-\delta^{(i)}} \in W, \tag{3.15}$$

where

$$
\begin{aligned}
u_i(a) =&\epsilon(d^{\beta-\delta^{(1)}},a)^{-1}\epsilon(d^{\beta-\delta^{(1)}},b)\epsilon(\partial_1'(b),\partial_1'(a))^{-1}\cdot\\
&\cdot\epsilon(d^{\beta-\delta^{(i)}},a)^{-1}\epsilon(d^{\beta-\delta^{(i)}},b)\epsilon(\partial_1'(b),\partial_i'(a))\\
=&\epsilon(d^{\delta^{(1)}-\delta^{(i)}},a)\epsilon(\partial_1',\partial_i'\partial_1')\in F.
\end{aligned}
\tag{3.16}
$$

Fix $a\in H(A)$ such that $\partial_1'(a)\neq 0$; then $\partial_i''=\partial_1'(a)\partial_i'-u_i(a)\partial_i'(a)\partial_1'\in \mathrm{Der}_F^{\epsilon}(A)$, $i=2,\dots,n$ are A-linear independent derivations satisfying the conditions of the Lemma, thus by induction $Ad^{\beta-\delta^{(n)}}\subset W$. ∎

LEMMA 3.8 (Z2, Corollary 2.13): *Suppose that A is a Γ-graded simple associative algebra of characteristic not 2, $\dim_{Z(A)}A>4$, and $\dim_{Z(A)}A\neq 8$. Then $[A,A]/([A,A]\cap Z)$ is a simple ϵ-Lie color algebra.* ∎

Our second main result in this section is the following.

THEOREM 3.9: *Suppose that F is of characteristic not 2, A is a Γ-graded D-simple (ϵ,Γ)-commutative associative F-algebra, where $D\subset Der^{\epsilon}(A)$ is color commutative and nonzero. Let $W=[A[D],A[D]]$.*

(i) *If $|\mathcal{J}|=\infty$, then $W=A[D]$.*

(ii) *Suppose $|\mathcal{J}|<\infty$. Let $\gamma\in\mathcal{J}$ be the maximal element of $\mathcal{J}$ (i.e., $|\gamma|=h(\mathcal{J})$). Then $W=\bigoplus_{\alpha\in\mathcal{J}\setminus\{\gamma\}}Ad^{\alpha}\oplus\mathcal{D}(A)d^{\gamma}$.*

(iii) *The Lie color algebra $\overline{A[D]}=W/F_1$ is simple except when $A=F_1[t]/(t^2-\lambda)$, where $\lambda\in F_1$ homogeneous, $D\subseteq F_1\frac{d}{dt}$ and both t and $\frac{d}{dt}$ have colors in Γ_- (note that if $\lambda\neq 0$, then $2\bar{t}=\bar{\lambda}$).*

Proof: If $\dim\mathcal{D}=1$ and $\mathcal{D}=\mathcal{D}_-$, we see that $|\mathcal{J}|=2$. Clearly (ii) is true in this case. Now suppose that $\mathcal{D}\neq\mathcal{D}_-$ if $\dim_{F_1}\mathcal{D}=1$. Then $|\mathcal{J}|>2$.

Consider any $\beta\in\mathcal{J}$ with $|\beta|\geq 2$. Using inductive assumption, we may suppose that $Ad^{\alpha}\subset W$ for all $\alpha\in\mathcal{J}(\beta)$ with $|\alpha|\leq|\beta|-2$. Assume that $\mathrm{supp}(\beta)=\{1,2,\dots,n\}$. Then for all $x,b\in H(A)$ we have $[xd^{\beta},b]\in W$ and

$$
\begin{aligned}
[xd^{\beta},b]=&x[d^{\beta},a]\\
\equiv&x\sum_{i=1}^{n}\beta_i\epsilon(d^{\beta-\delta^{(1)}},b)\epsilon^{+}(\delta^{(i)},\beta-\delta^{(i)})^{-1}d_i(b)d^{\beta-\delta^{(i)}}\ (\mathrm{mod}\,W).
\end{aligned}
\tag{3.17}
$$

By Lemma 3.4, $d_1,d_2,\dots,d_n$ are A-linear independent derivations. By Lemma 3.7, $Ad^{\beta-\delta^{(i)}}\subset W$ for $i=1,2,\dots,n$. This in particular proves (i), and that $\bigoplus_{\alpha\in\mathcal{J}\setminus\{\gamma\}}Ad^{\alpha}\subset W$ if $|\mathcal{J}|<\infty$. Next we assume that $|\mathcal{J}|<\infty$. Then $|\mathcal{I}|=m<\infty$ and either $\mathrm{char}\,F=p>0$ (in this case $\gamma_i=p-1$ or 1 for $i\in\mathcal{I}$) or

$\mathcal{D} = \mathcal{D}_-$ (in this case $\gamma_i = 1$ for $i \in \mathcal{I}$). Then $\mathcal{D}(A)d^\gamma = [\mathcal{D}, Ad^\gamma] \subset W$. Thus $W' = \bigoplus_{\alpha\in\mathcal{J}\setminus\{\gamma\}} Ad^\alpha \oplus \mathcal{D}(A)d^\gamma \subset W$. To prove $W \subset W'$, let $\alpha, \beta \in \mathcal{J}$ be such that $\alpha+\beta-\gamma \in \mathcal{J}\setminus\{0\}$. It suffices to show the coefficient of d^γ in $[xd^\alpha, yd^\beta]$ to be in $D(A)$. By (2.7) and (2.10), the coefficient of d^γ in $[ud^\alpha, vd^\beta]$ for $u, v \in H(A)$ is

(3.18)

$$\begin{aligned}&\binom{\alpha}{\gamma-\beta}\epsilon^+(\alpha+\beta-\gamma,\gamma-\beta)^{-1}\epsilon(d^{\gamma-\beta},v)\epsilon^+(\gamma-\beta,\beta)ud^{\alpha+\beta-\gamma}(v)\\ &-\epsilon(ud^\alpha,vd^\beta)\binom{\beta}{\gamma-\alpha}\epsilon^+(\alpha+\beta-\gamma,\gamma-\alpha)^{-1}\epsilon(d^{\gamma-\alpha},u)\epsilon^+(\gamma-\alpha,\alpha)vd^{\alpha+\beta-\gamma}(u)\\ &\qquad=\epsilon^+(\alpha+\beta-\gamma,\gamma-\beta)^{-1}\epsilon(d^{\gamma-\beta},v)\epsilon^+(\gamma-\beta,\beta)(\binom{\alpha}{\gamma-\beta}ud^{\alpha+\beta-\gamma}(v)\\ &\qquad\quad-\binom{\beta}{\gamma-\alpha}\epsilon(\bar{u},\beta+\alpha-\gamma)d^{\alpha+\beta-\gamma}(u)v).\end{aligned}$$

Noting that $\mathrm{char}F = p > 0$ or $\gamma_i = 1$, from the condition $\alpha, \beta, \gamma-\alpha-\beta \in \mathcal{J}\setminus\{0\}$ we see that

$$\binom{\alpha}{\gamma-\beta} = \binom{\alpha}{\alpha+\beta-\gamma} = (-1)^{|\alpha+\beta-\gamma|}\binom{\beta}{\gamma-\alpha}. \tag{3.19}$$

Applying d_i to $u'v'$ for any $u'v' \in H(A)$, we have

$$d_i(u')v' = -\epsilon(u', d_1)u'd_i(v') \in D(A)).$$

Using this and (3.19), one can easily deduce that the right-hand side of (3.18) is in $D(A)$. This proves (ii).

Now we prove (iii) by using Lemma 3.8. So we assume that $\dim_{F_1} A[D] \leq 4$ or $\dim_{F_1} A[D] = 8$, A is Γ-graded D-simple.

Case 1: $\dim_{F_1} A[D] = 8$.

From Theorem 3.2, we see that $\dim_{F_1} A = 2$ or 4 and $\dim_{F_1} F_1[D] = 4$ or 2 respectively.

Consider $\dim_{F_1} A = 2$ and $\dim_{F_1} F_1[D] = 4$. From the definition we see that $A[D] \subset \mathrm{End}(A)$, i.e., $8 = \dim_{F_1} A[D] \leq \dim_{F_1} \mathrm{End}(A) = 4$. a contradiction. Thus this subcase does not occur.

Now suppose $\dim_{F_1} A = 4$ and $\dim_{F_1} F_1[D] = 2$ as operators on A. Then $\dim_{F_1} F_1D = 1$, and let $F_1D = F_1\partial$. If $\bar{\partial} \in \bar{F}_1$, we may assume that $\bar{\partial} = 0$, and in this case

$$\partial^2 = a\partial + b \quad\text{and}\quad \bar{a} = \bar{b} = 0. \tag{3.20}$$

Applying (3.20) to 1 we deduce that $b = 0$. Applying (3.20) to xy for $x, y \in A$, we deduce that $\partial(x)\partial(y) = 0$ for $x, y \in A$, i.e., $\partial(A)\partial(A) = 0$. We can easily see that $A = AA = (A\partial(A))(A\partial(A)) = 0$, which is impossible. Thus $\bar{\partial} \notin \bar{F}_1$. Then

$$\partial^2 = b, \quad \text{and} \quad 2\bar{b} = \bar{\partial} \text{ if } b \neq 0. \tag{3.21}$$

Applying (3.21) to 1 we deduce that $\partial^2 = 0$. Then $\partial(A) = F_1$, which implies that $\dim_{F_1} \ker(\partial) = 3$, contrary to $F_1 = \ker(\partial)$. Therefore this subcase does not occur either.

CASE 2: $\dim_{F_1} A[D] \leq 4$.

From Theorem 3.2, we see that $\dim_{F_1} A = 2$ and $\dim_{F_1} F_1[D] = 2$. Suppose $A = F_1 \oplus F_1 x$ and $D = F\partial$. We can choose x such that $x^2 = \lambda \in F_1$. We divide the discussion into two subcases.

SUBCASE 1: $\bar{x} \in \Gamma_+$.

Applying ∂ to $x^2 = \lambda$, we deduce that $x\partial(x) = 0$. Then $\lambda\partial(x) = x^2\partial(x) = 0$, which implies $\lambda = 0$ and $\partial(x) = ax$ for some $a \in F_1^*$. So $F_1 x$ is a D-ideal of A, contrary to the D-simplicity of A. So this subcase does not occur.

SUBCASE 2: $\bar{x} \notin \Gamma_+$.

Then $\bar{x} \notin \bar{F}_1 = \{\bar{a} | a \in F_1\}$, and we can choose $\partial \in AD$ such that $\partial(x) = x$ or 1. If $\partial(x) = x$ and $\lambda = 0$, then $F_1 x$ is a D-ideal of A, contrary to the D-simplicity of A. Thus, if $\partial(x) = x$ we always have $\lambda \neq 0$, and then replacing ∂ by $\lambda^{-1}x\partial$, we can always choose ∂ such that $\partial(x) = 1$. Then we can easily verify that A is D simple, but $\overline{A[D]}$ is not a simple Lie color algebra. These are the exceptions in (iii). This has proved (iii). ∎

Example 3.10: Let $n \geq 2$ and let $x_1, x_2, \ldots, x_n$ be n symbols which have colors in Γ_-. Let A be the free ϵ-commutative associative algebra generated by $x_1, x_2, \ldots, x_n$. Clearly A has dimension 2^n. Let D be the space spanned by the derivations $\partial_i = \partial/\partial x_i, i = 1, 2, \ldots, n$. Then we obtain a simple Lie color algebra $\overline{A[D]}$ of dimension $2^{2n} - 2$. In particular, if we take $\Gamma = \mathbb{Z}/2\mathbb{Z}$, $\epsilon(i,j) = (-1)^{ij}, i, j \in \mathbb{Z}/2\mathbb{Z}$, then $A = \Lambda^n(x_1, x_2, \ldots, x_n)$ is the exterior algebra and we obtain the simple Hamiltonian Lie superalgebra $\overline{A[D]} = H(2n)$ (see §3.3 in [K]).

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